

# Three Qubit GHZ Correlations and Generalized Bell Experiments

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Received April 9, 2003

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We present a brief historical introduction to the topic of Bell's theorem. Next we present the surprising features of the three-particle Greenberger–Horne–Zeilinger (GHZ) states. Finally we shall present a method of analysis of the GHZ correlations, which is based on a numerical approach, which is effectively equivalent to the full set of Bell inequalities for correlation functions for the given problem. The aim of our numerical approach is to answer the following question: Do additional possible local settings lead for the GHZ states to more pronounced violation of local realism (measured by the resistance of the quantum nature of the correlations with respect to “white” noise admixtures)?

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## 1. INTRODUCTION: THE EARLY HISTORY OF THE PROBLEM

In the Introduction part of this paper we give a picture to the readers, especially the young ones, of the years when the term *quantum information* was not yet invented; however some basic research, that later gave birth to this new branch of physics, already began.

In 1964, Bell (1964) demonstrated that no local and realistic (that is classical relativistic) theory could ever agree with all predictions of quantum mechanics. His theorem showed that the idea of Einstein, *et al.* (1935) (EPR) of completing quantum mechanics, so that the resulting theory would be deterministic, is impossible. The theorem of Bell raises the following profound question: Can one model natural phenomena with a local relativistic theory? Moreover, the theorem provided a blueprint for an experimental test of this problem.

The consequences of the Bell theorem are so dramatic, so that experiments were needed. But there were two problems. The first one was that the original

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Bell's inequality required from the quantum two-particle system to possess perfect correlations. This is possible in theory, and indeed such entangled two-particle states exist in the Hilbert Space, however in the laboratory this is impossible. Simply in every experiment some noise is inevitable. The second problem was to find a source of entangled states that could give rise to observable quantum effects, that could be used in a Bell test. Both problems were solved in the trail-blazing paper by Clauser *et al.* (1969), the famous CHSH paper. A new type of Bell inequality was proposed, which did not require the perfect correlations. It gives a bound of correlations describable by local realistic theories. But this is not all. The authors noticed that the two-photon cascades in calcium result in emissions of pairs of photons with entangled polarizations, i.e., a source for the pairs of entangled particles needed for a test of Bell's inequality (now rather the CHSH one) was found, and the two-photon emissions, already an interesting phenomenon, were shown to be a very exotic effect in this case. Had this effect been known to Einstein and his colleagues, most probably the EPR paper would have been completely different. Since now a lot of experimental and theoretical physics is devoted to studying entangled states (usually in the form of entangled polarizations), more, even a whole new branch of physics emerged (quantum information) which tries to understand and exploit as a resource entanglement, this pinpointing by CHSH of the first controllable source of entanglement deserves to be called a great discovery. Yes, earlier entangled states were present in both theory and experiment but their drastically nonclassical properties were never observed directly. The new source enabled precisely this.

In few years time the actual experiment was performed by Freedman and Clauser (1972). To the amazement of many, Bell's inequalities were violated by a natural phenomenon observed in the lab. The prediction of the existence of entangled states of light was experimentally confirmed.

Another important step was the paper by Clauser and Horne (1974) in which they derived yet another Bell-type inequality, now customarily called the CH one. This inequality has several very important features. It is testable (like the CHSH one), it implies the CHSH one (but the CHSH does not imply the CH one), and is perfect for the analysis of the threshold parameters required for a "loophole-free" Bell test.

The dramatic consequences of Bell's theorem, and the falsification of local realism in the experiments of Clauser, caused a reaction in camp of researchers who were sceptical about the universal validity or completeness of quantum theory. This reaction resulted in many papers in which the above-mentioned "loopholes," i.e., imperfections of the Freedman–Clauser test of Bell inequalities, were studied, and which according to their authors could invalidate the experiment as a falsification of local realism.

Aspect *et al.* (1982) designed and performed Bell experiments aimed at closing on of the loopholes. The same calcium cascade was the source, but much more

effective pumping (by lasers) was used, and therefore the statistics were now much better. But especially important was the experiment in which the polarizations to be measured at two faraway detection stations were set effectively during the flight of the photons. This guaranteed that the measurement setting at side A was absolutely unknown at side B at the moment of the detection of the photon (and vice versa). Thus any spooky theory that could “explain” the quantum correlations via a local and realistic model, employing the “loophole” of the Clauser experiment (namely fixed polarizer settings throughout the each experimental run) was closed. In this way it was for the first time experimentally established that realistic theories of nature must be necessarily nonlocal. This was the most important loophole to be closed. Other loopholes are associated only with imperfections in the measuring devices (like detector efficiency). However since the quantum predictions are so well reproduced already in an imperfect experiment, why should we expect some deviations in a more precise one?

To summarize, the above experimental tests of local realism falsified this idea (so cherished by, e.g., Einstein), and in that way solved the Einstein–Bohr debate (almost) definitely in favor of Bohr. The existence in nature of entangled states was experimentally confirmed. Much later, it was experimentally proved that entanglement can be utilized directly in quantum cryptography (the protocol of Ekert, 1991). Quantum cryptography is now already within the realm of applied physics. Entanglement is essential in the process of quantum teleportation, and in various quantum communication and quantum information schemes. The very topic of entanglement leads to such surprises like the ultra-non-classical Greenberger *et al.* (1989) correlations. But in the beginning of all that were the ideas of Bell, and the early experiments. They showed with new strength, how strange is the quantum world, and that this strangeness can be experimentally observed. Now we begin to benefit from that.

## 2. SUMMARY

Further down in the paper we shall present a brief introduction to the Bell theorem, which will lead us to the simplest. Bell inequality (as far as the derivation is concerned), which is the aforementioned CHSH one. Next we shall give a brief introduction to the surprising features of the three-particle-Greenberger–Horne–Zeilinger (Greenberger *et al.*, 1989) (GHZ) states. Finally we shall present a new method of analysis of the GHZ correlations, which is based on a numerical approach, which is effectively equivalent to the full set of Bell inequalities for correlation functions for the given problem.<sup>5</sup> Such a set is known in the case of

<sup>5</sup>The set of Bell inequalities is “full” when it constitutes the sufficient and necessary condition for the existence of local realistic model for the given process, for the given number of local settings for each of the observers.

three qubits in the case of experiments involving two alternative settings for each observer (Weinfurter and Żukowski, 2001; Werner and Wolf, 2001; Żukowski and Brukner, 2002). In this case the full set can be expressed in the form of a single generalized inequality, and therefore it is easy to analyze. The full set of inequalities for three settings per observer can be in principle found using the method presented by Pitovsky and Svozil (2001). However, one can expect an enormous number of them.

The aim of our numerical approach is to answer the following question: Do additional possible local settings lead, in the case of three-qubit GHZ states, to more pronounced violations of local realism (measured by the resistance of the quantum nature of the correlations to “white” noise admixtures)?

### 3. BELL THEOREM

Before the advent of Bell (1964) theorem, despite Einstein’s doubts, the question of the existence of a more detailed description of individual events in the microworld than the probabilistic one provided by quantum mechanics was treated as interesting, however not falsifiable, and therefore as irrelevant as the question of “how many angels fit on the tip of the needle.” In early sixties, Bell (1966)<sup>6</sup> conjectured that if there is any conflict between quantum mechanics and the *realistic theories*,<sup>7</sup> it may be confined to *local*<sup>8</sup> versions of such theories. This led him to formulate his famous theorem of profound scientific and philosophical consequences.

#### 3.1. Bell Inequality

Consider pairs of particles (say, photons) simultaneously emitted in well-defined opposite directions. After some time the photons arrive at two very distant measuring devices A and B operated by two characters: Alice and Bob. Their apparatuses have a knob which specifies, which dichotomic (i.e., two-valued, yes–no, 0–1, one bit) observable they actually measure.<sup>9</sup> One can assign to the two possible results the numbers +1 (for *yes*, bit value one) and –1 (for *no*, bit value nil).<sup>10</sup> Alice and Bob are at any time (also in a “delayed choice” mode, after an

<sup>6</sup> Bell (1966) was written before Bell (1964).

<sup>7</sup> Realism, the cornerstone of classical physics, is a view that any physical system (i.e., also a subsystem of a compound system) carries full information (deterministic or probabilistic) on results of *all* possible experiments that can be performed upon it.

<sup>8</sup> A theory is local if it assumes that information cannot travel faster than light.

<sup>9</sup> For example for a device consisting of a polarizing beam splitter and two detectors behind its outputs, this knob would specify the orientation of the polarizer; if the device is a Mach–Zehnder interferometer (plus two detectors at the two exits) the knob would set the phase shift, etc. The photon may be registered only behind one or the other output ports of such devices.

<sup>10</sup> We assume perfect situation in which the detectors never fail to register a photon.

emission) *both free* to independently choose the observables (knob settings) that they want to measure. Their choice is absolutely independent of the workings of the source, and can be done at any time.

Let us assume that each photon pair carries full information (deterministic or probabilistic) on the values of the results of all possible experiments that can be performed on it<sup>11</sup> (*realism*). Also, by *locality*, choices made by them which are simultaneous in certain reference frame cannot influence each other (in Alice’s region of space-time, which contains the measurement event, there is no information whatsoever available on Bobs choice, and vice versa); the choice made on one side cannot influence the results on the other side.

For simplicity, assume that Alice chooses to measure either observable  $\hat{A}_1$  or  $\hat{A}_2$ , and Bob either  $\hat{B}_1$  or  $\hat{B}_2$ . Let us denote the hypothetical results that they may get for the  $j$ -th pair by  $A_1^j$  and  $A_2^j$ , for Alices two possible choices, and  $B_1^j$  and  $B_2^j$ , for Bobs. The numerical values of these results (+1 or -1) are defined by the two eigenvalues of the observables. Since, always either  $|B_1^j - B_2^j| = 2$  and  $|B_1^j + B_2^j| = 0$ , or  $|B_1^j - B_2^j| = 0$  and  $|B_1^j + B_2^j| = 2$ , the following relation holds

$$A_1^j B_1^j + A_1^j B_2^j + A_2^j B_1^j - A_2^j B_2^j = A_1^j (B_1^j + B_2^j) + A_2^j (B_1^j - B_2^j) = \pm 2. \quad (1)$$

Imagine now that  $N$  pairs of photons to are emitted pair by pair ( $N$  is sufficiently large,  $\sqrt{1/N} \ll 1$ ). The average value of the products of the local values for a joint test (often called the Bell correlation function) during which, for all photon pairs, only one pair of observables, say  $\hat{A}_n$  and  $\hat{B}_m$ , is chosen by the local observers is given by

$$E(A_n, B_m) = \frac{1}{N} \sum_{j=1}^{j=N} A_n^j B_m^j, \quad (2)$$

where  $n = 1, 2$ , and  $m = 1, 2$ . The relation implies that for the four *possible* choices of pairs of observables the following “Bell” inequality must be satisfied

$$-2 \leq E(A_1, B_1) + E(A_1, B_2) \quad (3)$$

$$+E(A_2, B_1) - E(a_2, B_2) \leq 2 \quad (4)$$

(Clauser *et al.*, 1969). In the actual experiment only in part of the cases (say, approximately 1/4-th) the given pair of observables would be measured, however, if  $N$  is very large, the correlation function obtained on a randomly preselected subensemble<sup>12</sup> of emissions cannot differ too much from the one that would have

<sup>11</sup> Note that in the present discussion, only this idea openly goes beyond “what is speakable” in quantum mechanics.

<sup>12</sup> The subensemble is selected by the choice of observables made by Alice and Bob *before* the actual measurements.

been obtained for the full ensemble<sup>13</sup> Therefore for the values of the *actually* chosen measurements the inequality also must hold.<sup>14</sup>

### 3.2. Bell Theorem Without Inequalities: Three Entangled Particles or More

If there are  $N > 2$  maximally entangled quantum systems (qubits), measurements on which are performed in  $N$  spatially separated regions by  $N$  independent observers, the correlations obtained in such experiment violate bounds imposed by local realism much stronger than in the case of two particles. Proofs of such violations can be limited to the experiments for which the perfect EPR-type correlations occur. That is, the EPR notion of elements of reality becomes self-contradictory.

As the simplest example, take a GHZ (Greenberger *et al.*, 1989) state of  $N = 3$  particles (Fig. 1):

$$|\psi(3)\rangle = \frac{1}{\sqrt{2}}(|a\rangle|b\rangle|c\rangle + |a'\rangle|b'\rangle|c'\rangle), \tag{5}$$

where  $\langle x|x'\rangle = 0(x = a, b, c, \text{ and kets denoted by one letter pertain to one of the particles})$ . The observers, Alice, Bob, and Cecil, measure the observables:  $\hat{A}(\phi_A), \hat{B}(\phi_B), \hat{C}(\phi_C)$ , defined by

$$\hat{X}(\phi_X) = |+, \phi_X\rangle\langle +, \phi_X| - |-, \phi_X\rangle\langle -, \phi_X| \tag{6}$$

and

$$|\pm, \phi_X\rangle = \frac{1}{\sqrt{2}}(\pm i|x'\rangle + \exp(i\phi_X)|x\rangle). \tag{7}$$

where  $X = A, B, C$ . The quantum prediction for the expectation value of the product of the three local observables is given by

$$E(\phi_A, \phi_B, \phi_C) = \langle \psi | \hat{A}(\phi_A) \hat{B}(\phi_B) \hat{C}(\phi_C) | \psi \rangle = \sin(\phi_A + \phi_B + \phi_C). \tag{8}$$

Therefore, if  $\phi_A + \phi_B + \phi_C = \pi/2 + k\pi$ , quantum mechanics predicts perfect correlations. For example, for  $\phi_A = \pi/2, \phi_B = 0, \text{ and } \phi_C = 0$ , whatever may be the results of local measurements of the observables, for say the particles belonging to the  $i$ -th triple represented by the quantum state  $|\psi(3)\rangle$ , they have to satisfy

$$A^i(\pi/2)B^i(0)C^i(0) = 1, \tag{9}$$

where  $X^i(\phi)$ ,  $X = A, B, \text{ or } C$  is the value of a local measurement of the observable  $\hat{X}(\phi)$  that *would have been* obtained for the  $i$ -th particle triple if the setting of the measuring device is  $\phi$ . By locality  $X^i(\phi)$  depends solely on the local parameter. The

<sup>13</sup> With  $N \rightarrow \infty$  the difference must approach zero, otherwise we would suspect that these two magnitudes pertain to two different physical processes (i.e., a systematic error must be involved).

<sup>14</sup> The presented Bell-type argument avoids any explicit introduction of hidden variables. All *local* hidden variable theories satisfy Bell inequalities.

Equation (9) indicates that we can predict with certainty the result of measuring the observable pertaining to one of the particles (say  $c$ ) by choosing to measure suitable observables for the other two. Hence the value  $X^i(\phi)$  are EPR elements of reality.

However, if the local apparatus settings are different, one *would have had*, e.g.

$$A^i(0)B^i(0)C^i(\pi/2) = 1, \quad (10)$$

$$A^i(0)B^i(\pi/2)C^i(0) = 1, \quad (11)$$

$$A^i(\pi/2)B^i(\pi/2)C^i(\pi/2) = -1. \quad (12)$$

Since  $X^i(\phi) \pm 1$ , if one multiplies side by side the Eqs. (9–12), the result is

$$A^i(\pi/2)B^i(\pi/2)C^i(\pi/2) = +1, \quad (13)$$

which contradicts (12). Thus the EPR elements of reality program break down. We have a “Bell theorem without inequalities” (Greenberger *et al.*, 1989).

#### 4. MORE THAN TWO SETTINGS PER EACH OBSERVER

The beautiful argument of the GHZ paper cannot be directly applied to experimental results. This is just like in the case of the original Bell inequality. The reason is exactly the same. One cannot observe perfect correlations in the lab. Some noise is inevitable. Therefore one must use Bell inequalities of a new type (for the first ones, see Mermin, 1990). In the standard approach to multiqubit Bell inequalities one assumes that each observer measures two randomly chosen dichotomic observables. The results of measurements are used to compute the set of correlations functions that one needs to check if there is a violation of certain Bell inequalities.

A possible extension to this scenario is that each observer measures more than two local observables. Obviously it cannot yield worse violation of local realism than in the standard case. However, it is difficult to find analytically optimal Bell inequalities for more than two local observables and one must resort to numerical methods.

There is a computationally efficient method of finding the optimal violation of local realism for arbitrary number of observers and measured observables (Żukowski *et al.*, 1999). It is based on well-known linear optimization algorithms. The greatest advantage of this method is that it gives necessary and sufficient conditions for the existence of local realistic description. The method has been successfully applied to the problem of violation of local realism for entangled pairs of q-Nits ( $N = 2, 3, \dots, 16$ ) and the numerical results of Kaszlikowski *et al.* (2000) have been later confirmed analytically by Kaszlikowski *et al.* (2001) and Collins *et al.* (2001).

In this paper we show an application of the mentioned numerical method to the GHZ correlations, i.e., to the maximally entangled state of three qubits.

### 5. DESCRIPTION OF THE METHOD

Let us consider the GHZ state of three qubits

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_2|0\rangle_3 + |1\rangle_1|1\rangle_2|1\rangle_3), \tag{14}$$

where  $|i\rangle_j$  is the  $i$ -th state of the  $j$ -th qubit.

Each observer measures the dichotomic observable  $\vec{n} \cdot \vec{\sigma}$ , where  $n = a, b, c$  ( $a$  for the first observer,  $b$  for the second one, and  $c$  for the third one),  $\vec{n}$  is a unit vector characterizing the observable which is measured by observer  $n$  and  $\vec{\sigma}$  is a vector the components of which are standard Pauli matrices. This family of observables  $\vec{n} \cdot \vec{\sigma}$  covers all possible dichotomic observables for a qubit system.

The probability of obtaining the result  $m = \pm 1$  for the observer  $a$ , when measuring the observable characterized by the vector  $\vec{a}$ , the result  $l = \pm 1$  for the observer  $b$ , when measuring the observable characterized by the vector  $\vec{b}$  and the result  $k = \pm 1$  for the observer  $c$ , when measuring the observable characterized by the vector  $\vec{c}$  is equal to

$$P_{QM}(m, l, k; \vec{a}, \vec{b}, \vec{c}) = \frac{1}{8}(1 + mla_3b_3 + mka_3c_3 + lkb_3c_3 + mlk \sum_{r,p,s=1}^3 M_{rps}a_r b_p c_s), \tag{15}$$

where  $a_r, b_p, c_s$  are components of vectors  $\vec{a}, \vec{b}, \vec{c}$  and where nonzero elements of the tensor  $M_{rps}$  are  $M_{111} = 1, M_{122} = -1, M_{212} = -1, M_{221} = -1$ . In spherical coordinates vectors  $\vec{a}, \vec{b}, \vec{c}$  read

$$\vec{n} = (\cos \phi_n \sin \theta_n, \sin \phi_n \sin \theta_n, \cos \theta_n), \tag{16}$$

where  $0 \leq \theta_n \leq \pi$  and  $0 \leq \phi_n \leq 2\pi$ . From now on we will be considering only the measurement of the observables characterized by vectors with the zero third component, which is equivalent to putting  $\theta_n = \pi/2$ . Thus, the formula (15) acquires simpler form (we have replaced  $\phi_a, \phi_b, \phi_c$  by  $\alpha, \beta, \gamma$ , respectively)

$$P_{QM}(m, l, k; \alpha, \beta, \gamma) = \frac{1}{8}(1 + mlk \sum_{r,p,s=1}^3 M_{rps}a_r b_p c_s) \tag{17}$$

in which only the term responsible for three qubit correlations is present.

The probabilities of obtaining one of the results in the local stations reveal no dependence on the local parameters,  $P_{QM}(l|\alpha) = P_{QM}(m|\beta) = P_{QM}(n|\gamma) = \frac{1}{2}$ .



Similarly, the probabilities describing two-qubit correlations do not reveal dependence on the local parameters, i.e.,  $P_{QM}(l, m | \alpha, \beta) = P_{QM}(m, n | \beta, \gamma) = P_{QM}(l, n | \alpha, \gamma) = \frac{1}{4}$ .

If there is a white noise in the quantum channel distributing qubits to the observers, we must replace the above quantum probabilities (17) by

$$P_{QM}^V(m, l, k | \alpha, \beta, \gamma) = \frac{1}{8}(1 + mlkV \sum_{r,p,s=1}^3 M_{rps} a_r b_p c_s), \tag{18}$$

where  $1 - V(0 \leq V \leq 1)$  is the amount of noise in the channel. The parameter  $V$  is often called “visibility.”

Let us define the correlation function  $E_{QM}^V(\alpha, \beta, \gamma)$  as

$$E_{QM}^V(\alpha, \beta, \gamma) = \sum_{m,l,k=-1}^1 mlk P(m, l, k; \alpha, \beta, \gamma) V \sum_{r,p,s=1}^3 M_{rps} a_r b_p c_s.$$

It equals  $V \cos(\alpha + \beta + \gamma)$ . As we see in the considered experiment, there is no single and two-qubit interference and the correlation function, which depends only on three-particle correlations, contains all information about correlations in the system.

In the experiment observer  $a$  chooses between  $N_a$  settings of the measuring apparatus  $\alpha_1, \dots, \alpha_{N_a}$ , observer  $b$  between  $N_b$  settings  $\beta_1, \dots, \beta_{N_b}$  and, finally, observer  $c$  between  $N_c$  settings  $\gamma_1, \dots, \gamma_{N_c}$ . For each triple of local settings we calculate the quantum correlation function  $E_{QM}^V(\alpha_i, \beta_j, \gamma_k)$  where  $i = 1, \dots, N_a, j = 1, \dots, N_b, k = 1, \dots, N_c$ . Thus we have a “tensor”  $Q_{ijk}(V) = E_{QM}^V(\alpha_i, \beta_j, \gamma_k)$  of quantum predictions.

Within the local hidden variables formalism the correlation function must have the following structure:

$$E_{LHV}(\alpha_i, \beta_j, \gamma_k) = \int d\rho(\lambda) A(\alpha_i, \lambda) B(\beta_j, \lambda) C(\gamma_k, \lambda), \tag{19}$$

where for dichotomic measurements

$$\begin{aligned} A(\alpha_i, \lambda) &= \pm 1, \\ B(\beta_j, \lambda) &= \pm 1, \\ C(\gamma_k, \lambda) &= \pm 1, \end{aligned} \tag{20}$$

and they represent the values of local measurements predetermined by the local hidden variables, denoted by  $\lambda$ , for the specified local settings. This expression is an average over a certain LHV distribution  $\rho(\lambda)$  of certain factorizable “tensors,” namely those with elements given by  $T_{ijk}(\lambda) = A(\alpha_i, \lambda) B(\beta_j, \lambda) C(\gamma_k, \lambda)$ . Since the only possible values of  $(A(\alpha_i, \lambda), B(\beta_j, \lambda), C(\gamma_k, \lambda))$  are  $\pm 1$ , there are only  $2^{N_a}$  different sequences of the values of  $(A(\alpha_1, \lambda), \dots, A(\alpha_{N_a}, \lambda))$ ,  $2^{N_b}$  different

sequences of  $(B(\beta_1, \lambda), \dots, B(\beta_{N_b}, \lambda))$ ,  $2^{N_c}$  different sequences of  $(C(\gamma_1, \lambda), \dots, C(\gamma_{N_c}, \lambda))$ , and consequently they form only  $2^{N_a+N_b+N_c}$  tensors  $T_{ijk}(\lambda)$ .

Therefore the structure of LHV models of  $E_{LHV}(\alpha_i, \beta_j, \gamma_k)$  reduces to discrete probabilistic models involving the average of all the  $2^{N_a+N_b+N_c}$  tensors  $T_{ijk}(\lambda)$ . In other words, the local hidden variables can be replaced, without any loss of generality, by a certain triple of variables  $l, m, n$  that have integer values respectively from  $1, \dots, 2^{N_a}, 1, \dots, 2^{N_b}, 1, \dots, 2^{N_c}$ . To each  $l$  we ascribe one possible sequence of the possible values of  $A(\alpha, l)$ , denoted from now on by  $A(\alpha, l)$ , similarly we replace  $B(\beta_j, \lambda)$  by  $B(\beta_j, m)$  and  $C(\gamma_k, \lambda)$  by  $C(\gamma_k, n)$ . With this notation the possible LHV models of the correlation function  $E_{LHV}(\alpha_i, \beta_j, \gamma_k)$  acquire the following simple form:

$$E_{LHV}(\alpha_i, \beta_j, \gamma_k) \sum_{l=1}^{2^{N_a}} \sum_{m=1}^{2^{N_b}} \sum_{n=1}^{2^{N_c}} p_{lmn} A(\alpha_i, l) B(\beta_j, m) C(\gamma_k, n), \tag{21}$$

with, of course, the probabilities satisfying  $p_{lmn} \geq 0$  and

$$\sum_{l=1}^{2^{N_a}} \sum_{m=1}^{2^{N_b}} \sum_{n=1}^{2^{N_c}} p_{lmn} = 1.$$

Please note that not all tensors  $T_{ijk}(lmn)$  are different (in fact only half of them differ).

The conditions for local hidden variables to reproduce the quantum prediction with a visibility  $V$  can be simplified to the problem of maximizing a parameter  $V$  for which exists a set of  $2^{N_a+N_b+N_c-1}$  probabilities  $p_{lmn}$ , such that

$$\sum_{l=1}^{2^{N_a}} \sum_{m=1}^{2^{N_b}} \sum_{n=1}^{2^{N_c}-1} \tilde{p}_{lmn} A(\vec{a}_i, l) B(\vec{b}_j, m) C(\vec{b}_k, n) = Q_{ijk}(V). \tag{22}$$

Because, for the given local settings (22) imposes linear constraints on the probabilities and the visibility, and we are looking for the maximal  $V$ , the problem can be solved by means of linear programming methods of optimization.

We want to find such local settings for which this maximal  $V$  reaches its minimum. This is due to the fact that in such a case the noise admixture,  $1 - V$ , is maximal. In such a case the nonclassical matrix of quantum correlations reveals the strongest resistance with respect to white noise admixtures. This can be treated as a measure of the “strength” of violation of local realism.

The set of linear equations (22) constitute a certain region in a  $D = 2^{N_a+N_b+N_c-1} + 1$  dimensional real space— $2^{N_a+N_b+N_c-1}$  probabilities plus the visibility. The border of the region consists of hyperplanes each defined by one of the equations belonging to (22), thus, if the equations do not contradict each other, the region is a convex set with a certain number of vertices. On this convex set we

define a linear function (cost function)  $f(p_1, \dots, p_{2N_a + N_b + N_{c-1}}, V) = V$ , and we seek its maximum.

The fundamental theorem of linear programming states that *the cost function reaches its maximum at one of the vertices*. Hence, it suffices to find numerical values of the cost function calculated at the vertices and then pick up the largest one. Of course, the algorithmic implementation of this simple idea is not so easy, for we must have a method of finding the vertices, for which the value of the function continually increases, so that the program reaches the optimal solution in the least possible number of steps. Calculating the value of the cost function at every vertex would take too much time, as there may be too many of them.

There are lots of excellent algorithms which solve the above optimization problem. Here we have used the algorithm invented the Gondzio (1995) and implemented in the commercial code HOPDM 2.30 (Higher Order Primal-Dual Method) written in C programming language.

However, finding the maximal visibility for the given local settings of the measuring apparatus is not enough. We should remember that our main goal is to find such local setting for which the threshold visibility is the lowest one. The maximal visibility  $V^{\max}$  returned by the HOPDM 2.30 procedure depends on the local settings entering right-hand side of (22). Thus, returned  $V^{\max}$  can be treated as the many variable function, which depends on  $N_a + N_b + N_c$  angles in the coplanar case and two times more in the noncoplanar one, i.e.,  $V^{\max} = V^{\max}(\vec{a}_1, \dots, \vec{a}_{N_a}, \vec{b}_1, \dots, \vec{b}_{N_b})$ .

Hence, we should also have a numerical procedure which finds the minimum of  $V^{\max}$ . Because we do not know much about the structure of  $V^{\max}$  as a function of the local settings, the only reasonable method of finding the  $V^{\max}$  minimum is the Downhill Simplex Method (DSM) (Nelder and Mead, 1965). The way it works toward finding the extremum is the following. If the dimension of the domain of a function is  $Dim$ , the DSM randomly generates  $Dim + 1$  points.

This way it creates a starting simplex with vertices being the points. Then it calculates the value of a function at the vertices and starts exploring the space by stretching and contracting the simplex. In every step if it finds a vertex where the value of the function is lower than in others, it “goes” in this direction.

We have checked four cases:  $N_a = N_b = N_c = 2, 3, 4, 5$ , with the result that the threshold visibility admitting local hidden variable model is  $V = \frac{1}{2}$ . This result is in concurrence with the threshold visibility obtained earlier in Mermin (1990) with the usage of appropriate Bell inequalities.

Because of the complexity of the space being the domain of the  $V^{\max}$  function to find a global minimum for the case  $N_a = N_b = N_c = 2, 3$  we have run the amoeba procedure 30 times with varied starting points. For  $N_a = N_b = N_c = 4$ , we calculated  $V^{\max}$  on 9000 randomly chosen sets of the local settings whereas in the case  $N_a = N_b = N_c = 5$  we have calculated  $V^{\max}$  on the following set of the local settings:  $\alpha_1 = 0, \alpha_2 = \pi/8, \alpha_3 = \pi/4, \alpha_4 = 3\pi/8, \alpha_5 = \pi/2$ ,

$\beta_1 = \gamma_1 = -\pi/4$ ,  $\beta_2 = \gamma_2 = -\pi/8$ ,  $\beta_3 = \gamma_3 = 0$ ,  $\beta_4 = \gamma_4 = -\pi/8$ ,  $\beta_5 = \gamma_5 = -\pi/4$ . In both cases the only reason for abandoning the DSM method was the exploding computational time.

An interesting feature of the results is that the threshold visibility  $V^{\max} = \frac{1}{2}$  is always achieved for such settings of the measuring apparatus which include as a subset the settings giving maximal violation of the inequalities. Similar result was obtained for two maximally entangled qubit, also using the numerical method, by Żukowski *et al.* (1999) and Massar *et al.* (2002).

## 6. CONCLUSIONS

The presented numerical approach to the three-qubit GHZ correlations gives the sufficient and necessary conditions for the existence of local hidden variables for the given experimental situation, i.e., for the fixed number of positions of the measuring apparatus at each side of the experiment.

For the cases of  $N_a = N_b = N_c = 2, 3$ , we have found such numerical values of the local settings for which the critical visibility admitting local hidden variables has the lowest possible value. Up to the possibility that the DSM procedure has not succeeded in finding the global minimum of  $V^{\max}$  the visibility  $V = \frac{1}{2}$  is the ultimate limit drawing the borderline between local hidden variables and quantum mechanics for these cases, i.e., for 2 and 3 settings of the measuring apparatus at each side of the experiment.

For  $N_a = N_b = N_c = 4$  the critical visibility returned by the program for every random choice of local settings has been always higher than  $\frac{1}{2}$ .

In the last case, i.e., for  $N_a = N_b = N_c = 5$ , we have found the threshold value for local settings including as a subset setting giving maximal violation of a three-particle Bell-type inequality with the result again  $V^{\max} = \frac{1}{2}$  (the DSM has not been used).

Unfortunately, because of the computer time and memory limitations, we could not check more settings of the measuring apparatus. Nevertheless, one could possibly conjecture that increasing the number of settings will not lead to a critical visibility lower than  $V = \frac{1}{2}$ . This is quite surprising especially when one considers the fact that already three settings per observation site lower the critical visibility for four GHZ qubits or more (Żukowski and Kaszlikowski, 1997).

The important aspect of the presented analysis of the GHZ correlations is that the numerical approach can be directly applied to measurement data.

## ACKNOWLEDGMENTS

DK acknowledges NUS Grant No. WBS: R-144-000-089-112. MZ acknowledges KBN Grant No. PBZ KBN 043/P03/2001.

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